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Double Bäcklund Transformations and Fission Solutions of Axisymmetric Gravitational Equations

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The results of Kyriakopoulos are extended to a double-complex form, and a double Bäcklund transformation of the Ernst equation is derived concretely. By using the noncommutative relation between this double Bäcklund transformation and the dual mapping, a fission-type generation process of new solutions is discussed.

1. INTRODUCTION AND PRELIMINARIES

Kyriakopoulos (1987) has given a method to find Bäcklund transformations (BT) of the Ernst equation. However, only ordinary complex functions are used in the method, and therefore half of the complete results are lost. In fact, in a series of papers Zhong (1985, 1988, 1990) has given a double-complex-function method combining ordinary complex numbers with hyperbolic complex numbers, and by using this method the solutions for the gravitational field are always obtained in pairs. It should be possible to apply the double-complex-function method to find BTs of the Ernst equation. The purpose of this paper is to extend Kyriakopoulos's results by the double-complex-function method. We find that there is a noncommutative relation between the double BT and the dual mapping defined by equation (8) below. By using this relation, a fission-type generation process of new solutions of the Ernst equation is described.

For the sake of convenience, some necessary results and symbols (Zhong, 1985, 1988, 1990) will be collected here. Let J denote the double imaginary unit, i.e., $J=i$ ($i^2=-1$) or $J=\varepsilon$ ($\varepsilon^2=1$, $\varepsilon\neq 1$). Let all a_n be real

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numbers, and let the series $\sum_{n=0}^{\infty} |a_n|$ be convergent; then

$$a(J) = \sum_{n=0}^{\infty} a_n J^{2n} \tag{1}$$

is called a double-real number. If $a(J)$ and $b(J)$ are both double-real numbers, $Z(J) = a(J) + J \cdot b(J)$ is called a double-complex number. Sometimes $Z(J)$ may be directly written as $Z(J) = (Z_C, Z_H)$, where

$$Z_C = Z(J = i), \qquad Z_H = Z(J = \varepsilon) \tag{2}$$

Let the metric of the axisymmetric stationary vacuum fields be

$$ds^2 = f(dt - \omega\, d\theta)^2 - f^{-1}[e^{\Gamma}(dz^2 + d\rho^2) + \rho^2\, d\theta^2] \tag{3}$$

where (ρ, z, θ) denote the cylindrical coordinates, and f, ω, and Γ are real functions of ρ and z only. We consider the double-complex Ernst equation (Zhong, 1985)

$$\mathrm{Re}(\mathscr{E}(J))\, \nabla^2 \mathscr{E}(J) = \nabla \mathscr{E}(J) \cdot \nabla \mathscr{E}(J) \tag{4}$$

where $\mathscr{E}(J) = \mathscr{E}(\rho, z; J) = F(\rho, z; J) + J \cdot \Omega(\rho, z; J)$ is a double-complex function; we have the operators $\nabla^2 = \partial_\rho^2 + \rho^{-1}\, \partial_\rho + \partial_z^2$ and $\nabla = (\partial_\rho, \partial_z)$. From a solution $\mathscr{E}(J)$ of equation (4), two different physical (real) solutions (f, ω) and $(\hat{f}, \hat{\omega})$ can be obtained as follows:

$$\begin{aligned} (f, \omega) &= (F_C, V_{F_C}^{-1}(\Omega_C)) \\ (\hat{f}, \hat{\omega}) &= (T(F_H), \Omega_H) \end{aligned} \tag{5}$$

where the NK transformations (Neugebauer and Kramer, 1969) are

$$\begin{aligned} T:&\quad f \to T(f) = \frac{\rho}{f} \\ V_f:&\quad \omega \to \varphi = V_f(\omega) \\ \varphi &= \int \frac{f^2}{\rho} (\partial_z \omega\, d\rho - \partial_\rho \omega\, dz) \end{aligned} \tag{6}$$

Let symbol "$\circ$" denote the operation of substitution about the imaginary units, i.e.,

$$\circ: \quad J \to \mathring{J}, \qquad \mathring{i} = \varepsilon, \quad \mathring{\varepsilon} = i \tag{7}$$

Then the dual mapping $d(J)=(d_C, d_H)$ is defined as (Zhong, 1990)

$$d(J):\quad \mathscr{E}(J)=F(J)+J\cdot\Omega(J)\to\hat{\mathscr{E}}(\mathring{J})=\hat{F}(\mathring{J})+\mathring{J}\cdot\hat{\Omega}(\mathring{J})$$
$$\hat{F}(\mathring{J})=T(F(J)) \tag{8}$$
$$\partial_\rho\hat{\Omega}(\mathring{J})=\frac{J^2\rho}{F^2(J)}\,\partial_z\Omega(J),\qquad \partial_z\hat{\Omega}(\mathring{J})=-\frac{J^2\rho}{F^2(J)}\,\partial_\rho\Omega(J)$$

By the dual mapping $d(J)$, a double-complex potential $\mathscr{E}(J)$ may be changed into another $\hat{\mathscr{E}}(J)$. However, although $\mathscr{E}(J)=(\mathscr{E}_C, \mathscr{E}_H)$ and $\hat{\mathscr{E}}(J)=(d_H(\mathscr{E}_H), d_C(\mathscr{E}_C))$ are different in form, both correspond to the same pair of gravitational field solutions in essence, i.e., $\mathscr{E}(J)$ and $\hat{\mathscr{E}}(J)$ are equivalent in the presence of gravitational fields.

2. DOUBLE BTs

Now we take the double-complex numbers

$$\xi(J)=\rho+J\cdot z,\qquad \eta(J)=\rho-J\cdot z \tag{9}$$

as independent variables; therefore equation (4) can be written as

$$\begin{aligned}&[(\partial_{\xi(J)}+\partial_{\eta(J)})^2+J^2(\partial_{\xi(J)}-\partial_{\eta(J)})^2]\mathscr{E}(J)\\&=-\frac{1}{\rho}(\partial_{\xi(J)}+\partial_{\eta(J)})\mathscr{E}(J)+\frac{2}{\mathscr{E}(J)+\mathscr{E}^*(J)}\\&\quad\times\{[\partial_{\xi(J)}\mathscr{E}(J)+\partial_{\eta(J)}\mathscr{E}(J)]^2+J^2[\partial_{\xi(J)}\mathscr{E}(J)-\partial_{\eta(J)}\mathscr{E}(J)]^2\}\end{aligned} \tag{10}$$

where the asterisk denotes the double-complex conjugation. When $J=i$, equation (10) is just changed into the ordinary complex form

$$\partial_{\xi_C}\partial_{\eta_C}\mathscr{E}_C=-\frac{1}{4\rho}(\partial_{\xi_C}+\partial_{\eta_C})\mathscr{E}_C+2(\mathscr{E}_C+\mathscr{E}_C^*)^{-1}\partial_{\xi_C}\mathscr{E}_C\cdot\partial_{\eta_C}\mathscr{E}_C \tag{11}$$

where $\xi_C=\rho+iz$ and $\eta_C=\rho-iz$. When $J=\varepsilon$, equation (10) is changed into a new form of Ernst equation under the hyperbolic coordinates,

$$\begin{aligned}(\partial^2_{\xi_H}+\partial^2_{\eta_H})\mathscr{E}_H=&-\frac{1}{2\rho}(\partial_{\xi_H}+\partial_{\eta_H})\mathscr{E}_H+2(\mathscr{E}_H+\mathscr{E}_H^*)^{-1}\\&\times[(\partial_{\xi_H}\mathscr{E}_H)^2+(\partial_{\eta_H}\mathscr{E}_H)^2]\end{aligned} \tag{12}$$

where $\xi_H=\rho+\varepsilon z$ and $\eta_H=\rho-\varepsilon z$.

First, we discuss the problem of how to find a BT of equation (10), i.e., a double BT of the Ernst equation. In the following we can ignore the effect of the choice of integral constants, and consider a double BT as a mapping

$\mathscr{T}(J)$ from an old solution $\mathscr{E}(J)=F(J)+J\cdot\Omega(J)$ to a new solution $\mathscr{E}'(J)=F'(J)+J\cdot\Omega'(J)$. Now, suppose that a double BT of equation (10) is taken in the following form:

$$\begin{aligned}\mathscr{T}(J):\quad &\mathscr{E}(J)\rightarrow\mathscr{E}'(J)=\mathscr{T}(J)(\mathscr{E}(J))\\ &\partial_{\xi(J)}\mathscr{E}'(J)=\gamma(J)\partial_{\xi(J)}\mathscr{E}(J)\\ &\partial_{\eta(J)}\mathscr{E}'(J)=\delta(J)\partial_{\eta(J)}\mathscr{E}(J)\end{aligned}\tag{13}$$

where $\gamma(J)$ and $\delta(J)$ both are double-complex functions. From equation (10) and the integrability condition

$$\partial_{\xi(J)}\,\partial_{\eta(J)}\mathscr{E}'(J)=\partial_{\eta(J)}\,\partial_{\xi(J)}\mathscr{E}'(J)\tag{14}$$

we obtain

$$\begin{aligned}&\gamma(J)=\delta(J)\\ &\partial_{\mathscr{E}(J)}\gamma(J)+\gamma(J)\partial_{\mathscr{E}'(J)}\gamma(J)+\frac{2\gamma(J)}{\mathscr{E}(J)+\mathscr{E}^*(J)}-\frac{2\gamma(J)\delta(J)}{\mathscr{E}'(J)+\mathscr{E}'^*(J)}=0\\ &\partial_{\mathscr{E}^*(J)}\gamma(J)+\gamma^*(J)\partial_{\mathscr{E}'^*(J)}\gamma(J)=0\end{aligned}\tag{15}$$

Let $\gamma(J)$ be taken in the following form:

$$\gamma(J)=Z_1[2F(J),2F'(J)]+J\cdot Z_2[2F(J),2F'(J)]\tag{16}$$

where Z_i $(i=1,2)$ is a double-real function; then we obtain the equations of Z_i as follows:

$$\begin{aligned}&\frac{\partial Z_1}{\partial F(J)}=J^2\,\frac{Z_2^2-J^2Z_1^2}{F'(J)}\\ &\frac{\partial Z_1}{\partial F'(J)}=-\frac{1}{F(J)}+\frac{2Z_1}{F'(J)}\\ &\frac{\partial Z_2}{\partial F(J)}=J^2\left(\frac{Z_1^2-J^2Z_2^2}{Z_2F(J)}-\frac{Z_1^2+Z_2^2}{Z_2F'(J)}\right)+(1+J^2)\,\frac{Z_1Z_2}{F'(J)}\\ &\frac{\partial Z_2}{\partial F'(J)}=J^2\left(\frac{Z_1^2+J^2Z_2^2}{Z_2F'(J)}-\frac{Z_1}{Z_2F(J)}\right)\end{aligned}\tag{17}$$

In addition, the consistency condition for equation (17) is

$$Z_1^2 - J^2 Z_2^2 = \frac{F'^2(J)}{F^2(J)} = \|\gamma(J)\|^2 \tag{18}$$

Hence, by equations (17) and (18) a solution of equation (15) is

$$\begin{aligned} \gamma(J) &= \frac{F'(J)}{F(J)} g(J) \\ g(J) &= g_1(J) + J \cdot g_2(J) \\ &= 1 + k(J)F(J)F'(J) \\ &\quad + J \cdot \{J^2[2k(J)F(J)F'(J) + k^2(J)F^2(J)F'^2(J)]\}^{1/2} \end{aligned} \tag{19}$$

where $k(J)$ is a double-real constant which must be chosen so as to satisfy

$$J^2[2k(J)F(J)F'(J) + k^2(J)F^2(J)F'^2(J)] \geq 0 \tag{20}$$

Moreover, we easily verify that $g(J)$ satisfies the relation

$$g(J)g^*(J) = 1 \tag{21}$$

By equations (13) and (19) the concrete form of the double BT $\mathscr{T}(J)$ is

$$\begin{aligned} \partial_{\xi(J)}\mathscr{E}'(J) &= \frac{F'(J)}{F(J)} g(J)\partial_{\xi(J)}\mathscr{E}(J) \\ \partial_{\eta(J)}\mathscr{E}'(J) &= \frac{F'(J)}{F(J)} g(J)\partial_{\eta(J)}\mathscr{E}(J) \end{aligned} \tag{22}$$

When $J = i$, equation (22) is an ordinary BT $\mathscr{T}_C$ for the ordinary complex Ernst equation (11),

$$\begin{aligned} &\mathscr{T}_C\colon \quad \mathscr{E}_C \rightarrow \mathscr{E}'_C \\ &\partial_{\xi_C}\mathscr{E}_C = \frac{F'_C}{F_C} g_C\, \partial_{\xi_C}\mathscr{E}_C \\ &\partial_{\eta_C}\mathscr{E}_C = \frac{F'_C}{F_C} g_C\, \partial_{\eta_C}\mathscr{E}_C \end{aligned} \tag{23}$$

where

$$g_C = g(J = i) = g_{1C} + ig_{2C} = 1 + k_C F_C F'_C + i(-2k_C F_C F'_C - k_C^2 F_C^2 F_C'^2)^{1/2}$$

This is just the result of Kyriakopoulos (1987) and Omote and Wadati (1981). When $J=\varepsilon$, one has a BT $\mathscr{T}_H$ of the hyperbolic complex Ernst equation (12),

$$\mathscr{T}_H: \quad \mathscr{E}_H \to \mathscr{E}'_H$$

$$\partial_{\xi_H}\mathscr{E}'_H = \frac{F'_H}{F_H} g_H\, \partial_{\xi_H}\mathscr{E}_H \tag{24}$$

$$\partial_{\eta_H}\mathscr{E}'_H = \frac{F'_H}{F_H} g_H\, \partial_{\eta_H}\mathscr{E}_H$$

where

$$g_H = g(J=\varepsilon) = g_{1H} + \varepsilon g_{2H} = 1 + k_H F_H F'_H + \varepsilon(2k_H F_H F'_H + k_H^2 F_H^2 F'^2_H)^{1/2}$$

Notice that $\mathscr{T}_H$ is similar to $\mathscr{T}_C$ in form; however, $\mathscr{T}_H$ and $\mathscr{T}_C$, in fact, are two different BTs, i.e., $\mathscr{T}_H$ and $\mathscr{T}_C$ mean two different relations among the gravitational field solutions.

Furthermore, by using the dual mapping $d(J)$ in equation (8) and the BT $\mathscr{T}_H$ we can derive a new BT,

$$\hat{\mathscr{T}}_C = d_H \circ \mathscr{T}_H \circ d_C: \quad \mathscr{E}_C \to \hat{\mathscr{E}}'_C$$

$$\partial_{\xi_C}\hat{\mathscr{E}}'_C = \frac{\hat{F}'_C}{F_C}(g_{1H} + g_{2H})\partial_{\xi_C}\mathscr{E}_C + \frac{\hat{F}'_C}{2\rho}(1 - g_{1H} - g_{2H}) \tag{25}$$

$$\partial_{\eta_C}\hat{\mathscr{E}}' = \frac{\hat{F}'_C}{F_C}(g_{1H} - g_{2H})\partial_{\eta_C}\mathscr{E}_C + \frac{\hat{F}'_C}{2\rho}(1 - g_{1H} + g_{2H})$$

where

$$\begin{aligned} g_{1H} &= 1 + k_H F_H \hat{F}'_H \\ g_{2H} &= (2k_H F_H \hat{F}'_H + k_H^2 F_H^2 \hat{F}'^2_H)^{1/2} \end{aligned} \tag{26}$$

Evidently, $\mathscr{T}_C$ and $\hat{\mathscr{T}}_C$ are not the same, i.e., $\hat{\mathscr{T}}_C$ is indeed a new BT. Of course, in our method equations (11) and (12) are equal in status; we have another pair ($\mathscr{T}_H$, $\hat{\mathscr{T}}_H$) for equation (12), where the BT $\hat{\mathscr{T}}_H$ is obtained by the mapping $d(J)$ and $\mathscr{T}_C$,

$$\hat{\mathscr{T}}_H = d_C \circ \mathscr{T}_C \circ d_H: \quad \mathscr{E}_H \to \hat{\mathscr{E}}'_H$$

$$\partial_{\xi_H}\hat{\mathscr{E}}'_H = \frac{\hat{F}'_H}{F_H}(g_{1C}\, \partial_{\xi_H}\mathscr{E}_H - g_{2C}\, \partial_{\eta_C}\mathscr{E}_H) + \frac{\hat{F}'_H}{2\rho}(1 - g_{1C} + g_{2C}) \tag{27}$$

$$\partial_{\eta_H}\hat{\mathscr{E}}'_H = \frac{\hat{F}'_H}{F_H}(g_{2C}\, \partial_{\xi_H}\mathscr{E}_H + g_{1C}\, \partial_{\eta_H}\mathscr{E}_H) + \frac{\hat{F}'_H}{2\rho}(1 - g_{1C} - g_{2C})$$

where

$$g_{1C} = 1 + k_C F_C \hat{F}'_C$$
$$g_{2C} = (-2k_C F_C \hat{F}'_C - k_C^2 F_C^2 \hat{F}'^2_C)^{1/2} \tag{28}$$

3. FISSION GENERATION OF NEW SOLUTIONS

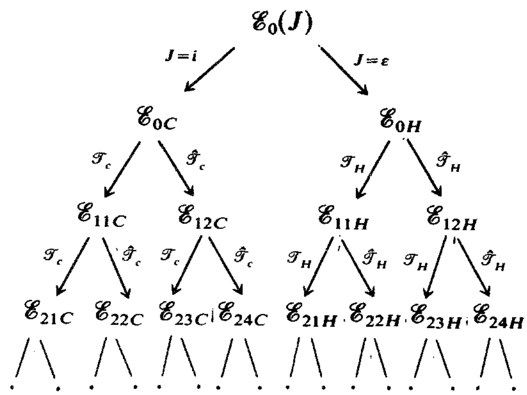
The dual mapping $d(J): \mathscr{E}(J) \rightarrow \mathscr{E}'(\mathring{J}) = d(J)(\mathscr{E}(J))$ is a special mapping; in the presence of gravitational fields it reflects an equivalence relation, but by it new solutions of the gravitational field cannot be generated. However, a change will take place if $d(J)$ is used with other transformations. We consider the following relations:

$$\begin{array}{ccc} \Sigma(J) & \xrightarrow{\mathscr{T}(J)} & \Sigma(J) \\ \downarrow{\scriptstyle d(J)} & & \downarrow{\scriptstyle d(J)} \\ \Sigma(J) & \xrightarrow{\hat{\mathscr{T}}(J)} & \Sigma(J) \end{array} \tag{29}$$

where $\Sigma(J)$ denotes the set consisting of all non-pure imaginary solutions of equation (10). If the above relation is noncommutative, i.e.,

$$\mathscr{T}(J) \neq \hat{\mathscr{T}}(J) = d(\mathring{J}) \circ \mathscr{T}(\mathring{J}) \circ d(J) \tag{30}$$

and $\mathscr{E}_0(J) = (\mathscr{E}_{0C}, \mathscr{E}_{0H}) = (\mathscr{E}_0(J=i), \mathscr{E}_0(J=\varepsilon))$ is a seed solution, then we have the generation process of new solutions as follows:



(31)

In this scheme, the new solutions are

$$\mathscr{E}_{11C} = \mathscr{T}_C(\mathscr{E}_{0C}), \qquad \mathscr{E}_{12C} = \hat{\mathscr{T}}_C(\mathscr{E}_{0C})$$
$$\mathscr{E}_{21C} = \mathscr{T}^2_C(\mathscr{E}_{0C}), \qquad \mathscr{E}_{22C} = \hat{\mathscr{T}}_C \circ \mathscr{T}_C(\mathscr{E}_{0C}) \tag{32}$$
$$\cdots$$

The above process looks like a nuclear fission reaction in form, and the generation rate of new solutions greatly exceeds that of the ordinary complex method. In fact, in our method 2^n ($n=1, 2, \ldots$) new solutions can be generated by the nth step, while in the ordinary complex method the corresponding process is only a single line on the left side in the scheme, i.e.,

$$\mathscr{E}_{0C} \to \mathscr{E}_{11C} \to \mathscr{E}_{21C} \to \ldots.$$

The key to the above discussion is the noncommutative relation (30). For the double BT derived in Section 2, the transformation $\hat{\mathscr{T}}(J)$ can be written as follows:

$$\begin{aligned}
\partial_{\xi(J)}\hat{\mathscr{E}}'(J) &= \frac{\hat{F}'(J)}{F(J)}\left[g_1(\mathring{J})\,\partial_{\xi(J)}\mathscr{E}(J) + \frac{1-J^2}{2}\,g_2(\mathring{J})\partial_{\xi(J)}\mathscr{E}(J)\right.\\
&\quad \left. - \frac{1+J^2}{2}\,g_2(\mathring{J})\partial_{\eta(J)}\mathscr{E}(J)\right]\\
&\quad + \frac{\hat{F}'(J)}{2\rho}\,[1 - g_1(\mathring{J}) + J^2 g_2(\mathring{J})]\;.\\
\partial_{\eta(J)}\hat{\mathscr{E}}'(J) &= \frac{\hat{F}'(J)}{F(J)}\left[\frac{1+J^2}{2}\,g_2(\mathring{J})\partial_{\xi(J)}\mathscr{E}(J)\right.\\
&\quad \left. + g_1(\mathring{J})\partial_{\eta(J)}\mathscr{E}(J) - \frac{1-J^2}{2}\,g_2(\mathring{J})\partial_{\eta(J)}\mathscr{E}(J)\right]\\
&\quad + \frac{\hat{F}'(J)}{2\rho}\,[1 - g_1(\mathring{J}) - J^2 g_2(\mathring{J})]
\end{aligned} \tag{33}$$

Evidently, the noncommutative relation generally holds for such $\hat{\mathscr{T}}(J)$ and $\mathscr{T}(J)$. This means that we indeed have a fission generation process for the stationary axisymmetric vacuum field solutions.

In addition, according to Zhong (1990), the dual mapping $d(J)$ indicates a symmetry of the set of stationary axisymmetric gravitational field solutions. Kyriakopoulos (1987) pointed out that in some cases the BT $\mathscr{T}_C$ in equation (23) will be included into Ehler's transformation. In our case the double BT $\mathscr{T}(J)$ in equation (22) can result, under a similar condition, in the double Ehler transformation (Zhong, 1985),

$$\mathscr{E}'(J) = \frac{a(J)\mathscr{E}(J) + J \cdot b(J)}{J \cdot c(J)\mathscr{E}(J) + d(J)} \tag{34}$$

where the double-real constants $a(J)$, $b(J)$, $c(J)$, and $d(J)$ satisfy

$$a(J)d(J) - J^2 b(J)c(J) = 1 \tag{35}$$

In this case the above fission generation process in fact corresponds to an infinite-dimensional double-complex group structure given by Zhong (1990).

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REFERENCES

Kyriakopoulos, E. (1987). *Journal of Physics A: Mathematical and General Physics*, **20**, 1667–1676.

Neugebauer, G., and Kramer, D. (1969). *Annalen der Physik*, **24**, 62–69.

Omote, M., and Wadati, M. (1981). *Journal of Mathematical Physics*, **22**, 961–964.

Zhong, Z.-Z. (1985). *Journal of Mathematical Physics*, **26**, 2589–2595.

Zhong, Z.-Z. (1988). *Scientia Sinica*, **A31**(4), 436–450.

Zhong, Z.-Z. (1990). *Science in China*, **A33**(9), 1089–1099.